On the Role of Source Terms in Continuum Traffic Flow Models

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Abstract

We introduce some models for vehicular traffic flow based on hyperbolic balance laws. We focus in particular on source terms for modeling highway entries and exits or local changes of the traffic flow due to inhomogeneities of the road. Rigorous well-posedness results and numerical investigations are presented. We show in particular how real phenomena (e.g. the formation of a queue) that are not captured by conservative systems are instead observable with our models.

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1 Introduction

This note is concerned with the role of source terms in traffic flow models based on hyperbolic systems of conservation laws. In the classical Lighthill-Whitham [17] and Richards [19] (LWR) model $\partial_t \rho + \partial_x \left(\rho \cdot v(\rho)\right) = 0$ a source term s is often introduced, leading to $\partial_t \rho + \partial_x \left(\rho \cdot v(\rho)\right) = s(t, x, \rho)$, to account for entries and exits to the considered road. Here, ρ is the car density and v is the traffic speed. In this case, a discussion of the source term can be developed also at an elementary level, see for instance [11, Chap. 3].

More recently, after the criticism by Daganzo [9], several 2×2 (or *"higher order"*) systems of conservation laws modeling traffic flow were proposed, see for instance [1, 5, 6]. All these models postulate the conservation of the number of vehicles and of another quantity related to traffic density and speed. In the case of fluid dynamics, the conservation of linear momentum justifies such a second conservation principle. In the case of traffic flow, no such principle seems to be present. Therefore, source terms need to be added, not only to account for entries and exits, but also to correct, when unjustified, the conservation of this second quantity.

More precisely, in the case of [5], the model is

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x \big(\rho \cdot v(\rho, q)\big) = 0 \\ \partial_t q + \partial_x \big((q - q_*) \cdot v(\rho, q)\big) = 0\end{array}\right. \qquad v(\rho, q) = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho}. \quad (1.1)$$

Here q is a sort of *weighted momentum* while R (the maximal car density) and q_* (related to wide jams [14]) are parameters that characterize the considered road, see also (3.7) below. The conservation of q, that is

$$\frac{d}{dt} \int_{\mathbb{R}} q(t, x)\, dx = 0\,,$$

is only partially justified, *a posteriori*, by the reasonably good predictions provided by the model. However, not only entries and exits may interfere with the conservation of q. The presence of a slope, for instance, may alter the net car flow, and hence also q.

In an experimental setting, the amount and the speed of cars entering a highway, for instance, may well be considered a measured, and hence given, data. From a modeling point of view, or also at a designing stage, a question naturally arises, namely that of choosing the most suitable source terms. We refer to [12] for a review of some balance laws commonly used to model traffic flow. In the present note, we state sufficient conditions that, if satisfied by the source terms, ensure that the model continues to enjoy reasonable requirements. For instance, the model (1.1), as also the models in [1, 6], displays the presence of *a priori* positive upper and lower bounds on traffic density and speed. Below, we select properties of the source terms s_ρ and s_q so that the system

$$\left\{\begin{array}{l}\partial_t \rho + \partial_x \big(\rho \cdot v(\rho, q)\big) = s_\rho(t, x, \rho, q) \\ \partial_t q + \partial_x \big((q - q_*) \cdot v(\rho, q)\big) = s_q(t, x, \rho, q)\end{array}\right. \qquad v(\rho, q) = \left(1 - \frac{\rho}{R}\right) \cdot \frac{q}{\rho} \quad (1.2)$$

keeps these good properties. The case of models with convective part different from (1.1) is also considered.

Our main result extends [7, Theorem 2.2]. Colloquially, it reads

Theorem 1.1 *Consider the balance law* $\partial_t u + \partial_x f(u) = s(t, x, u)$ *where*

(i) $f \colon \Omega \mapsto \mathbb{R}^n$ *generates a smooth and strictly hyperbolic Temple system;*

(ii) $s \colon [0, +\infty[\times \mathbb{R} \times \Omega \mapsto \mathbb{R}^n$ *is measurable, smooth in* u*, locally Lipschitz and sub-linear in* u *uniformly in* (t, x)*, with uniformly bounded variation in* x*;*

(iii) u_o *attains values in a subset* $\mathcal{U}$ *of* Ω *which is invariant both for the conservative law* $\partial_t u + \partial_x f(u) = 0$ *and for the differential equation* $\partial_t u = s(t,x,u)$*.*

Then the balance law $\partial_t u + \partial_x f(u) = s(t,x,u)$ *is well posed, i.e. it admits a unique solution defined for all* $t \in [0,+\infty[$ *that, over bounded time intervals, is a locally Lipschitz function of the initial data* u_o *with Lipschitz constant depending on* $\mathrm{TV}(u_o)$*.*

Remark that the above assumptions on f, respectively on s, are exactly those hypotheses that ensure the global well posedness of $\partial_t u + \partial_x f(u) = 0$, respectively of $\dot{u} = s(t,x,u)$. See Theorem 2.2 for a more precise statement.

The plan of the paper is the following. In Section 2 we introduce a system of conservation laws unifying models [1] and [5]. Then, we extend the recent result [7], to obtain the well-posedness of the initial value problem for these systems in the presence of source terms. Section 3 is devoted to some examples of right hand sides satisfying the posed conditions. In Section 4, we show some numerical integrations. In particular, we show how the introduction of a suitable source term in the second equation may lead to the formation of a queue, a phenomenon that can be described neither by the purely convective models [1, 5, 6], nor by the standard LWR model. Finally, Section 5 contains the analytical proofs. Part of the results of this paper were announced in [2].

2 Source Terms and Invariance

Let Ω be the closure of a non empty, open and connected subset of $\mathbb{R}^n$. We consider systems of conservation laws of Temple type, i.e. of the form

$$\partial_t u + \partial_x f(u) = 0 \tag{2.1}$$

where $f\colon \Omega \mapsto \mathbb{R}^n$ is smooth; $Df(u)$ admits n real distinct eigenvalues $\lambda_1(u) < \lambda_2(u) < \ldots < \lambda_n(u)$; shock and rarefaction curves coincide and there is a set of Riemann coordinates. Refer to [4, 8] for the basic definitions concerning (2.1) and to [3, 20] for specific results on Temple systems.

In particular if $n = 2$ and $u = (\rho, y)$, $f(u) = v(u)\,u$, where v is a scalar function, then (2.1) becomes the following 2×2 system, see [15]:

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x \big(\rho \cdot v(\rho,y)\big) = 0 \\ \partial_t y + \partial_x \big(y \cdot v(\rho,y)\big) = 0\,. \end{array} \right. \tag{2.2}$$

Lemma 2.1 *Let* $v = v(\rho,y)$ *be a smooth function in a domain* $\Omega \subseteq \mathbb{R}^2$*. If*

$$\rho\,\partial_\rho v + y\,\partial_y v \neq 0 \qquad \textit{for all } (\rho,y) \in \Omega\,, \tag{2.3}$$

then (2.2) *is a Temple system, the eigenvalues being* $\lambda_1 = v + \rho\,\partial_\rho v + y\,\partial_y v$*,* $\lambda_2 = v$*. The second characteristic field is linearly degenerate.*

System (2.2) comprises both the model (1.1) presented in [5] and the one in [1], namely

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t \left(\rho(v + p(\rho))\right) + \partial_x \left(\rho v(v + p(\rho)\right) = 0\,. \end{array}\right. \tag{2.4}$$

In the former case, $y = q - q_*$ and (2.3) is satisfied in $\{(\rho, q) \in \,]0, R] \times [0, +\infty[\, : (\rho, q) \neq (R, 0)\}$. In the latter case, $y = \rho v + \rho\, p(\rho)$ where, according to [1], the *pressure* $p \colon [0, +\infty[\, \mapsto \mathbb{R}$ is smooth and satisfies $p' > 0$. The left hand side in (2.3) reduces to $-\rho\, p'(\rho) < 0$, so that Lemma 2.1 applies. In both cases, $\lambda_1 < \lambda_2 = v$, so that no information travels faster than vehicles.

For the sake of completeness, we note that Payne model [18], where $p(\rho) \sim \rho^\gamma$, is not a Temple system and does not fit into (2.2).

Relevant requirements that a traffic model should meet can be described by means of *invariant* sets.

A set $\mathcal{U} \subseteq \Omega$ is invariant for (2.1) if the solution $u = u(t, x)$ corresponding to an initial data u_o with range in $\mathcal{U}$, i.e. $u_o(\mathbb{R}) \subseteq \mathcal{U}$, has range in $\mathcal{U}$, i.e. $u(t, \mathbb{R}) \subseteq \mathcal{U}$, as long as the solution exists. Essentially, see [13], $\mathcal{U}$ is invariant if and only if any Riemann problem with data in $\mathcal{U}$ admits a solution with range in $\mathcal{U}$.

Invariant sets need not be convex. Indeed, the invariant sets in (1.1) are

$$\mathcal{U} = \left\{(\rho, q) \in [0, R] \times \mathbb{R} \colon v(\rho, q) \in [V_1, V_2],\, q_* + \frac{q - q_*}{\rho} R \in [Q_1, Q_2]\right\} \tag{2.5}$$

where $0 \leq V_1 \leq V_2 \leq +\infty$ and $0 \leq Q_1 \leq Q_2 \leq +\infty$, see Figure 1.

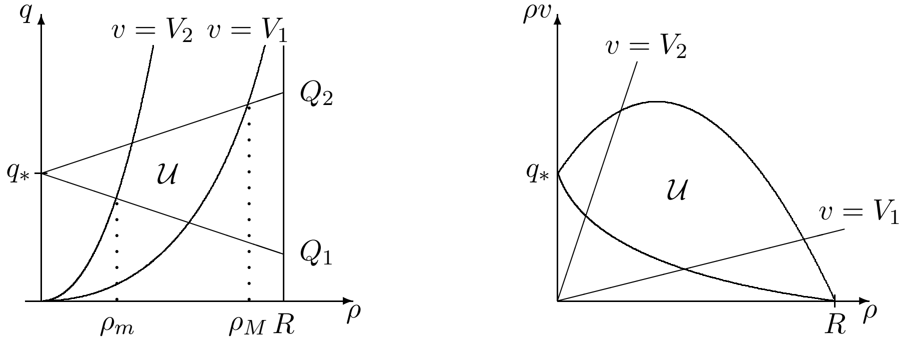

Figure 1: Invariant domain (2.5) for (1.1). Left, in the (ρ, q) plane. Right, in the $(\rho, \rho v)$ plane: the *fundamental diagram*.

The invariance of $\mathcal{U}$ in (2.5) ensures that the solutions to (1.1) have positive bounded density and speed. Moreover, if $V_1 = 0$, the traffic speed vanishes if and only if the traffic density reaches the maximum R.

In [7] it is proved that, under suitable conditions, invariance is *not* lost when adding source terms, i.e. passing from (1.1) to (1.2). Therefore, also the solutions to (1.2) have positive bounded density and speed.

Recall that a set $\mathcal{U}$ in $\mathbb{R}^n$ is *positively invariant* with respect to the ordinary differential equation with parameter x

$$\dot{u}(t) = s\left(t, x, u(t)\right) \tag{2.6}$$

if for any u_o in $\mathcal{U}$ and any $x \in \mathbb{R}$ the corresponding solution $u = u(t)$ attains values in $\mathcal{U}$. A classical sufficient condition for the positive invariance with respect to (2.6) of a set $\mathcal{U}$ with sufficiently regular boundary $\partial\mathcal{U}$ is

$$s(t, x, u) \cdot n(u) \geq 0 \tag{2.7}$$

for $t \geq 0$, $x \in \mathbb{R}$ and $u \in \partial\mathcal{U}$, $n(u)$ being the inner normal at u to $\mathcal{U}$.

To state Theorem 1.1 more formally, denote by $\mathbf{BV}(\mathbb{R}, \mathcal{U})$ the space of functions defined on $\mathbb{R}$ with values in $\mathcal{U}$ having bounded total variation. Denote by $\mathcal{R}(\mathcal{U})$ the set of Riemann data in $\mathcal{U}$, i.e. the set of the functions $u^l \chi_{]-\infty,0]}(x) + u^r \chi_{]0,+\infty[}(x)$ where the states u^l, u^r are in $\mathcal{U}$ and χ_A is the characteristic function of the set A. $\mathbf{L}^\infty_c(\mathbb{R})$ is the set of compactly supported measurable bounded functions in $\mathbb{R}$ and we define

$$\mathbf{X} = \bigcup_{u \in \mathcal{R}(\mathcal{U})} \left(u + \mathbf{L^1}(\mathbb{R}, \mathcal{U})\right) \cap \mathbf{BV}(\mathbb{R}, \mathcal{U}).$$

Theorem 2.2 *Assume that*

(i) $f\colon \Omega \mapsto \mathbb{R}^n$ *is smooth;* $Df(u)$ *admits* n *real strictly separated eigenvalues; shock and rarefaction curves of* (2.1) *coincide; there exists a system of smooth Riemann coordinates;*

(ii) $s\colon [0, +\infty[\times \mathbb{R} \times \Omega \mapsto \mathbb{R}^n$ *is measurable in* (t, x, u) *and smooth in* u*; there exists a positive finite measure* μ *on* $\mathbb{R}$ *such that for a.e.* $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \leq x_2$

$$\left| s(t, x_1^-, u) - s(t, x_2^+, u) \right| \leq \mu\left([x_1, x_2]\right)\ ; \tag{2.8}$$

there exist functions $A, B \in \mathbf{L^1_{loc}}\left([0, +\infty[; \mathbb{R}\right)$ *and for every compact set* $K \subseteq \mathcal{U}$ *there exist* $L_K \in \mathbf{L^1_{loc}}\left([0, +\infty[; \mathbb{R}\right)$ *such that for a.e.* $t \in [0, +\infty[$*, a.e.* $x \in \mathbb{R}$

$$\begin{aligned} \left| s(t, x, u_2) - s(t, x, u_1) \right| &\leq L_K(t)\, |u_2 - u_1| \quad \text{for } u_1, u_2 \in K, & (2.9)\\ \left| s(t, x, u) \right| &\leq A(t) + B(t)|u| \quad \text{for } u \in \Omega\,; & (2.10) \end{aligned}$$

(iii) there exists a domain $\mathcal{U} \subseteq \Omega$ *that is invariant both for the conservation law* (2.1) *and for the ordinary differential equation* (2.6).

Then, for every initial data $u_o \in \mathbf{X}$ the Cauchy problem

$$\left\{\begin{array}{l} \partial_t u + \partial_x f(u) = s(t,x,u) \\ u(0,x) = u_o(x) \end{array}\right. \tag{2.11}$$

admits a global solution $u\colon [0,+\infty[\, \times \mathbb{R} \mapsto \mathcal{U}$ with $u(t) \in \mathbf{X}$ for a.e. t. Moreover, for all $M, T > 0$ there exists a positive L such that if $u_o, u'_o \in \mathbf{X}$ with $\mathrm{TV}(u_o), \mathrm{TV}(u'_o) \leq M$, then the corresponding solutions u, u' satisfy

$$\left\|u(t) - u'(t)\right\|_{\mathbf{L}^1} \leq L \cdot \left\|u_o - u'_o\right\|_{\mathbf{L}^1} \tag{2.12}$$

for all $t \in [0,T]$.

The proof of the above result is deferred to Section 5. There, we provide further estimates on the $\mathbf{L}^\infty$ norm and on the total variation of the solution. Here, we only note that u_o and u'_o need to have the same limits at $\pm\infty$ for the right hand side of (2.12) to be finite.

3 Models

We consider now the case of the 2×2 traffic model (1.2). For simplicity we choose the domain (2.5) with $V_1 = 0$ and set $V_2 = V$, i.e. we let

$$\mathcal{U} = \left\{(\rho,q) \in [0,R] \times \mathbb{R}\colon v(\rho,q) \in [0,V],\ q_* + (q - q_*)\frac{R}{\rho} \in [Q_1, Q_2]\right\} \tag{3.1}$$

as in Figure 2. Denote by ρ_i the abscissas of the intersection points of the curve $v(\rho,q) = V$ with the lines $q = q_* + (Q_i - q_*)\rho/R$, for $i = 1,2$. Then, (2.7) holds if and only if for all (t,x) all the following conditions hold:

$$\begin{array}{lll} V s_\rho \geq \left(1 - \dfrac{\rho}{R}\right)^2 s_q & \text{where } v(\rho,q) = V\,, & \rho \in [\rho_1, \rho_2] \\ (Q_2 - q_*)s_\rho \geq R s_q & \text{where } \dfrac{q - q_*}{Q_2 - q_*} = \dfrac{\rho}{R}\,, & \rho \in [\rho_2, R] \\ (Q_1 - q_*)s_\rho \leq R s_q & \text{where } \dfrac{q - q_*}{Q_1 - q_*} = \dfrac{\rho}{R}\,, & \rho \in [\rho_1, R] \\ s_\rho(t,x,R,q) \leq 0 & \forall q \in [Q_1, Q_2]. & \end{array} \tag{3.2}$$

As a consequence, in the case of (1.2), (3.1) and condition (3.2) replace *(iii)* in Theorem 2.2. Concerning condition (3.2) remark in particular that

- if s_q is identically zero, then we can take as s_ρ any smooth positive function vanishing on the segment $\rho = R$, $q \in [Q_1, Q_2]$.
- if s_ρ is identically zero, then we can take as s_q any negative function vanishing on the segment $\frac{q - q_*}{Q_1 - q_*} = \frac{\rho}{R}$, $\rho \in [\rho_1, R]$.

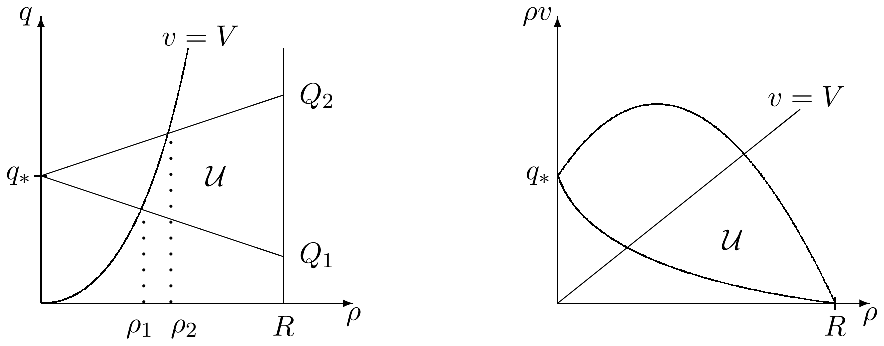

Figure 2: Invariant domain (3.1) for (1.1). Left, in the (ρ, q) plane and, right, in the $(\rho, \rho v)$ plane.

The choice $Q_2 = +\infty$ is admissible, leading to substitute (3.2) with

$$\begin{array}{ll} V s_\rho \geq \left(1 - \frac{\rho}{R}\right)^2 s_q & \text{for } v(\rho, q) = V \,,\ \rho \in [\rho_1, \rho_2] \\ (Q_1 - q_*) s_\rho \leq R s_q & \text{for } \frac{q - q_*}{Q_1 - q_*} = \frac{\rho}{R} \,,\ \rho \in [\rho_1, R] \\ s_\rho(t, x, R, q) \leq 0 & \forall q \in [Q_1, +\infty[\ . \end{array} \tag{3.3}$$

Analogous conditions can be obtained when $\mathcal{U}$ is given by (2.5).

3.1 Entries and exits

Assume that vehicles may exit the considered road along the interval $[a, b]$. More precisely, let $g_{\text{out}}(t)$ be the fraction of the traffic density per unit time that exits along $[a, b]$. Denoting $a_{\text{out}}(t, x) = g_{\text{out}}(t) \chi_{[a,\, b]}(x)$, we are lead to introduce the source term $a_{\text{out}}(t, x) \rho / R$. Similarly, if there is an entry along $[a, b]$, we set $a_{\text{in}}(t, x) = g_{\text{in}}(t) \chi_{[a,\, b]}(x)$ and model the entry through $a_{\text{in}}(t, x) \left(1 - \rho/R\right)$, $g_{\text{in}}(t)$ being the fraction of the available traffic density (per unit time) used by the vehicles entering the road.

Then a model for entries/exits having as convective part system (1.1) is

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x (\rho v) = a_{\text{in}}(t, x) \left(1 - \frac{\rho}{R}\right) - a_{\text{out}}(t, x) \frac{\rho}{R} \\ \partial_t q + \partial_x \left((q - q_*) v\right) = -\left(\frac{a_{\text{in}}(t, x)}{R} + \frac{a_{\text{out}}(t, x)}{R}\right) (q - q_*). \end{array} \right. \tag{3.4}$$

The source term in second equation is motivated by a sort of *"stabilizing"* effect that entries and exits have on the flow.

Lemma 3.1 *Let* $0 < Q_1 < Q_2$. *Then the following sets are invariant domains for* (3.4) *(i)* $\mathcal{U} = \big\{(\rho, q) \in [0, R] \times \mathbb{R}\colon q_* + (q - q_*)R/\rho \in [Q_1, Q_2]\big\}$*;*

(ii) $\mathcal{U} = \big\{(\rho, q) \in [0, R] \times \mathbb{R}\colon v \le V,\ q_* + (q - q_*)R/\rho \in [0, Q_2]\big\}$ *if* $a_{\mathrm{out}} = 0$ *(i.e., only one entry is present and no exit) and* $V \ge (\sqrt{5} - 2)q_*/R$*;*

(iii) $\mathcal{U} = \big\{(\rho, q) \in [0, R] \times \mathbb{R}\colon v \ge V,\ q_* + (q - q_*)R/\rho \in [Q_1, Q_2]\big\}$ *if* $a_{\mathrm{in}} = 0$ *(i.e., only one exit is present and no entries) and* $V \ge 0$*.*

Proof. We prove only the case when $a_{\mathrm{out}} = 0$, the other cases being straightforward consequences of (2.7). Along the curve $v = V$ the right hand side in (2.7) is positive iff

$$\rho \ge \rho_0 = R\frac{q_* - RV}{q_* + RV}.$$

If $q_* - RV \le 0$, then we have finished. If $q_* - RV > 0$ then we need that the intersection point of the line $q = q_* - \frac{q_*}{R}\rho$ with the curve $v = V$ lies on the right of ρ_0. The abscissa of the intersection point is

$$\rho_{\mathrm{int}} = R\frac{RV + 2q_* - \sqrt{RV(RV + 4q_*)}}{2q_*}.$$

A simple calculation shows that $\rho_{\mathrm{int}} \ge \rho_0$ iff $V \ge (\sqrt{5} - 2)q_*/R$. □

We point out that the condition above $V \ge (\sqrt{5} - 2)q_*/R$ is satisfied in many fundamental diagrams, for instance in [14]. The functions $g_{\mathrm{in}}(t)$, $g_{\mathrm{out}}(t)$ are bounded in (t, x) and have uniform bounded variation in x, hence Theorem 2.2 applies.

3.2 Relaxation terms

In several models, see for instance [12, 18], a relaxation term is added to the second equation. Keeping the same expression (1.2) for v, consider for instance the system

$$\begin{cases} \partial_t\rho + \partial_x(\rho v) = a\left(1 - \dfrac{\rho}{R}\right) \\ \partial_t q + \partial_x\big((q - q_*)v\big) = -\dfrac{a}{R}(q - q_*) + \dfrac{R}{R - \rho}\dfrac{\rho(v_e - v)}{\tau}. \end{cases} \tag{3.5}$$

Here $a = a(t, x) = g(t)\chi(x)$ as in (3.4), $v_e = v_e(\rho)$ is an equilibrium velocity and τ a relaxation time. The source terms describe a possible entry together with relaxation.

It is easy to check that if (3.5) is written for variables ρ and v then the term $\frac{R}{R-\rho}\frac{\rho(v_e-v)}{\tau}$ becomes the familiar $\frac{v_e-v}{\tau}$ (see e.g. [12, (2)]).

Lemma 3.2 *Assume* $0 \le Q_1 < Q_2$*; then:*

(i) the set (2.5) *with* $Q_1 > 0$, $V_1 = 0$ *and* $V_2 = +\infty$ *is an invariant domain for* (3.5) *if*

$$\left\{\begin{array}{rcl} v_e(\rho) & \geq & \left(q_* + (Q_1 - q_*)\frac{\rho + a\tau}{R}\right)\left(\frac{1}{\rho} - \frac{1}{R}\right) \\ v_e(\rho) & \leq & \left(q_* + (Q_2 - q_*)\frac{\rho + a\tau}{R}\right)\left(\frac{1}{\rho} - \frac{1}{R}\right); \end{array}\right. \tag{3.6}$$

(ii) if $a = 0$*, let* $0 < V_1 < V_2$ *and assume* $V_1 \leq v_e(\rho) \leq V_2$ *for any* $\rho \in [\rho_m, \rho_M]$ *and* (3.6)*; then* (2.5) *is an invariant domain for* (3.5)*.*

The proof is immediate by (2.7). When $a = 0$ the bounds (3.6) reduce to

$$\left(q_* + (Q_1 - q_*)\frac{\rho}{R}\right)\left(\frac{1}{\rho} - \frac{1}{R}\right) \leq v_e(\rho) \leq \left(q_* + (Q_2 - q_*)\frac{\rho}{R}\right)\left(\frac{1}{\rho} - \frac{1}{R}\right).$$

v_e must satisfy this estimate also when a is present, but then a bound independent of τ is required. In any case, if we choose $Q_1 < q_* < Q_2$ then

$$v_e^*(\rho) = \frac{q_*}{\rho}\left(1 - \frac{\rho}{R}\right) \tag{3.7}$$

is certainly an equilibrium velocity. This gives a further interpretation of the parameter q_*. Lemma 3.2 holds even in the case of an equilibrium velocity $v_e(\rho, q)$ depending also on q; see [10], where the conservative part is given by (2.4). Refer to [16] for related relaxation results on the Lighthill-Whitham and Richards model.

In *(ii)* of Lemma 3.2 we assumed $a = 0$ only for technical simplicity. The presence of entries can be dealt similarly. As in the previous case, Theorem 2.2 applies under the natural assumption that $g \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}$.

3.3 Changes in Traffic Speed

This paragraph and the following deal in particular with the case $s_\rho = 0$. The total amount of cars is then conserved; a source term in the second equation (1.2) allows to describe local changes in the road.

Assume that a descent along a highway is present between $x = x_1$ and $x = x_2$. A choice for the source term in the second equation is $s_q(t, x, \rho, q) = \chi(x) \cdot \rho \cdot a(t, \rho, q)$ where χ localizes the descent and $a \geq 0$ is the mean acceleration. Therefore the system reads

$$\left\{\begin{array}{l} \partial_t \rho + \partial_x \left(\rho \cdot v(\rho, q)\right) = 0 \\ \partial_t q + \partial_x \left((q - q_*) \cdot v(\rho, q)\right) = \chi(x) \cdot \rho \cdot a(t, \rho, q). \end{array}\right. \tag{3.8}$$

The conditions (3.2) are satisfied if $a(t, \rho, q) = 0$ where $v(\rho, q) = V$ and where $\frac{q - q_*}{Q_2 - q_*} = \frac{\rho}{R}$. The first requirement is quite a natural one: at the maximal speed the mean acceleration is zero. The second one is not really essential: in fact we still obtain an invariant domain if $Q_2 = +\infty$, see (2.5). In such a case the invariant domain becomes simply

$$\mathcal{U} = \left\{ (\rho, q) \in [0, R] \times \mathbb{R} \colon v \in [0, V] \, , \; q_* + (q - q_*) \frac{R}{\rho} \geq Q_1 \right\} .$$

Then Theorem 2.2 applies if, for instance, the acceleration a is bounded. The case of an ascent is analogous; however now $a \leq 0$. Conditions (3.2) are satisfied if $a(t, \rho, q) = 0$ where $\frac{q-q_*}{Q_1-q_*} = \frac{\rho}{R}$.

Remark that analogous source terms may be used to model different phenomena, for instance a fog bank or a narrowing/widening of the road.

The introduction of a source term as in (3.8) with $a < 0$ leads to the formation of a queue; this phenomenon can be seen very easily from numerical computations, see Case 1 in Paragraph 4.2 below. It is important to stress that such a behavior cannot be obtained in the case of a single scalar conservation law (as for instance in the LWR case), because of the maximum principle, [8].

3.4 A Geometric Term for the Second Equation

Assume that $s_\rho = 0$. Referring to the invariant domain (2.5) with $V_1 = 0$, we define the following source term for the second equation in (1.2):

$$s_q(t, \rho, q) = \left\{ \begin{array}{ll} \left(q_* - q - (q_* - Q_2) \dfrac{\rho}{R} \right) \cdot \theta(t, \rho, q) & \text{if } \theta \geq 0 \\ \left(q_* - q - (q_* - Q_1) \dfrac{\rho}{R} \right) \cdot \theta(t, \rho, q) & \text{if } \theta < 0 \, , \end{array} \right.$$

where θ is a smooth function ranging in $[-1, 1]$. The domain $\mathcal{U} = \{ (\rho, q) \in [0, R] \times \mathbb{R} \colon \; q_* + (q - q_*) \frac{R}{\rho} \in [Q_1, Q_2] \}$ is invariant for (1.2) if $s_\rho = 0$ and s_q as above. Roughly speaking, from a geometric point of view, the term s_q allows the maximum variation for q while mantaining the invariance of $\mathcal{U}$.

3.5 Source Terms for Another Convective Model

We consider now Aw-Rascle's model (2.4) with an entry term as in (3.4):

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x (\rho v) = a \left(1 - \dfrac{\rho}{R} \right) \\ \partial_t y_t + \partial_t (y v) = -a \dfrac{y}{R} . \end{array} \right. \tag{3.9}$$

A meaningful invariant domain must satisfy the condition $v \geq 0$ in order to rule out the negative speeds. But the source term points inside the region $v \geq 0$ iff $-Rp(\rho) - \rho(R - \rho)p'(\rho) \geq 0$, which never holds true if $p > 0$. The criterion (2.7) is therefore useless if $p > 0$.

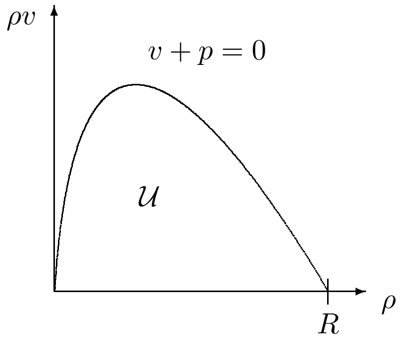

Figure 3: Invariant domain for system (3.9).

In the case $p(\rho) = P \ln \frac{\rho}{R}$ the pressure is negative. An invariant domain for the conservative part is $\mathcal{U} = \left\{(\rho, y) \colon v \geq 0,\ y \leq 0,\ \rho \in [0, R]\right\}$, see Figure 3. This domain is invariant also for (3.9), as it easily follows from (2.7). However, traffic speed is not bounded on $\mathcal{U}$.

4 Numerical Examples

This section is devoted to a few simple, albeit realistic, situations that do not fit into the previous analytical framework because no domain is invariant for the full system (1.2). Therefore, we base the following observations on short time numerical integrations.

4.1 Exits

In the case of an exit, see Paragraph 3.1, the presence of a source term in the second equation may lead to more realistic behaviors, as is shown by the following two examples.

Consider model (1.2) with the following parameters: $R = 150 \frac{\mathrm{vehicle}}{\mathrm{Km}}$, $q_* = 2.000 \frac{\mathrm{vehicle}}{\mathrm{h}}$, on a stretch long $L = 30\,\mathrm{Km}$, with constant initial data $\rho(0, x) = 80 \frac{\mathrm{vehicle}}{\mathrm{Km}}$ and $q(0, x) = 15.000 \frac{\mathrm{vehicle}}{\mathrm{h}}$, so that $v(0, x) = 87.5 \frac{\mathrm{Km}}{\mathrm{h}}$. There is an exit in the interval $[15\,\mathrm{Km}, 15.01\,\mathrm{Km}]$.

The data above, as well as those in the following, are deliberately chosen to emphasize phenomena that can be described through source terms and that occur also in realistic situations.

Case 1: Exit without slowdown. Let

$$\begin{aligned} s_\rho(t, x, \rho, q) &= \begin{cases} -0.25 \cdot \rho \cdot v(\rho, q) & x \in [15, 15.01] \\ 0 & \text{otherwise} \end{cases} \\ s_q(t, x, \rho, q) &= 0 \end{aligned} \tag{4.1}$$

with $v(\rho, q)$ as in (1.2). Then, at time $T = 3\,\mathrm{min}$ the solution of (1.2)-(4.1) is showed in Figure 4.

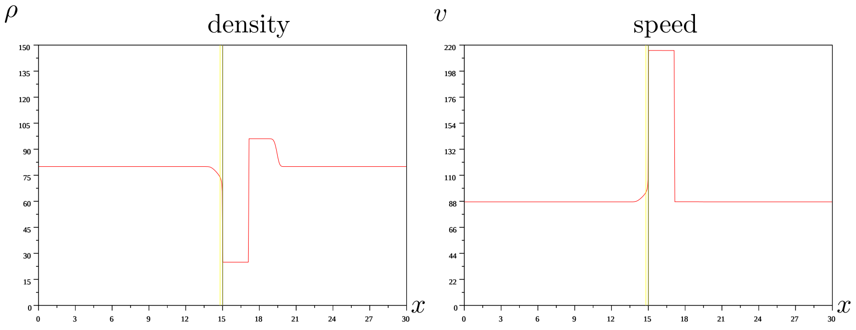

Figure 4: Traffic density ρ and speed v at $T = 3\,\mathrm{min}$.

Case 2: Exit with slowdown. Assume that 200 m before the exit, drivers slow down, i.e.

$$\begin{aligned} s_\rho(t,x,\rho,q) &= \begin{cases} -0.25 \cdot \rho \cdot v(\rho,q) & x \in [15, 15.01] \\ 0 & \text{otherwise} \end{cases} \\ s_q(t,x,\rho,q) &= \begin{cases} -5.000 \frac{\mathrm{Km}}{\mathrm{h}^2} \cdot \rho & x \in [14.8\,\mathrm{Km}, 15\,\mathrm{Km}] \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.2}$$

Then, at time $T = 3\,\mathrm{min}$ the solution of (1.2)-(4.2) is showed in Figure 5.

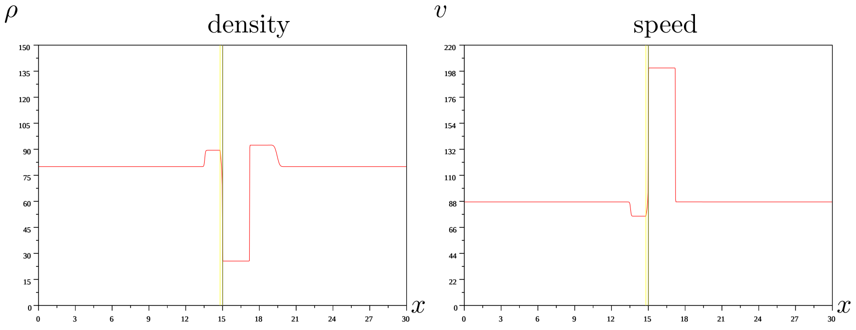

Figure 5: Traffic density ρ and speed v at time $T = 3\,\mathrm{min}$.

Note the qualitative difference before the entry. The presence of the source term in the second equation (realistically) causes an increase in the density and a decrease in the speed *before* the exit.

Obviously, whenever the source terms are independent from ρ and q, no non trivial invariance condition may hold. Hence, Theorem 2.2 can not be applied in these cases. On the other hand, source terms depending only on t and x naturally arise in the testing of these models against real traffic data, since measurement devices provide inflow, outflows and through flows dependent just on t and x.

4.2 Variations in the Traffic Speed

In the case of an ascent or a descent, for instance, the presence of a source term in the second equation may describe a mean variation in traffic speed, see Paragraph 3.3.

Consider model (1.2) with the following parameters: $R = 150 \frac{\mathrm{vehicle}}{\mathrm{Km}}$, $q_* = 2.000 \frac{\mathrm{vehicle}}{\mathrm{h}}$ on a stretch long $L = 30\,\mathrm{Km}$ with constant initial data $\rho(0,x) = 80 \frac{\mathrm{vehicle}}{\mathrm{Km}}$ and $q(0,x) = 18.000 \frac{\mathrm{vehicle}}{\mathrm{h}}$, so that $v(0,x) = 105 \frac{\mathrm{Km}}{\mathrm{h}}$.

Case 1: Decrease of the speed. In the stretch $[15\,\mathrm{Km}, 15.5\,\mathrm{Km}]$ the traffic speed decreases. Let

$$\begin{aligned} s_\rho(t,x,\rho,q) &= 0 \\ s_q(t,x,\rho,q) &= \begin{cases} -5.000 \frac{\mathrm{Km}}{\mathrm{h}^2} \cdot \rho & x \in [15\,\mathrm{Km}, 15.5\,\mathrm{Km}] \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.3}$$

Then, at time $T = 6\,\mathrm{min}$ the solution of (1.2)-(4.3) is given by Figure 6. The deceleration causes a queue to propagate backwards. The increase of the speed along the stretch corresponds to the decrease of the density there. Note that the total number of vehicles is conserved. Examples of decreasing of the speed could be ascent or fog banks. Remark that for the sake of simplicity all initial data are constant.

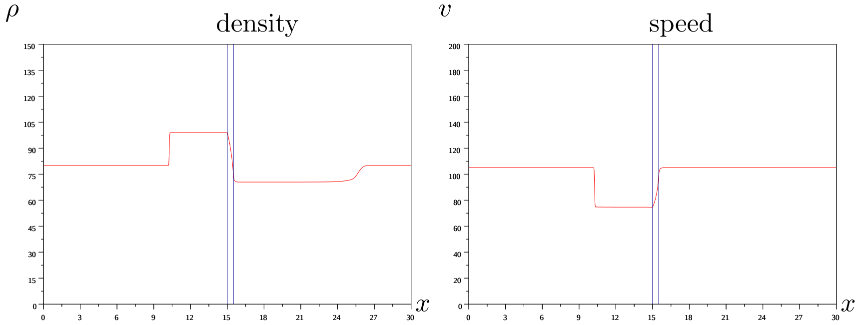

Figure 6: Traffic density ρ and speed v at $T = 6\,\mathrm{min}$.

Case 2: Increase in the speed. The traffic speed increases in the stretch $[9\,\mathrm{Km}, 9.5\,\mathrm{Km}]$. Let

$$\begin{aligned} s_\rho(t,x,\rho,q) &= 0 \\ s_q(t,x,\rho,q) &= \begin{cases} 5.000 \frac{\mathrm{Km}}{\mathrm{h}^2} \cdot \rho & x \in [9\,\mathrm{Km}, 9.5\,\mathrm{Km}] \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.4}$$

Then, at time $T = 7\,\mathrm{min}$ the solution of (1.2)-(4.4) is displayed in Figure 7. The acceleration causes a wave with higher speed and lower density to propagate forward. As in the previous case, the increase of the speed matches

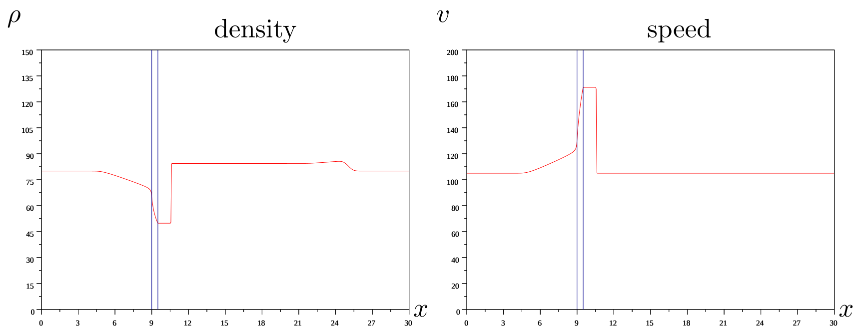

Figure 7: Traffic density ρ and speed v at $T = 7\,\mathrm{min}$.

the decrease of the density. Again, the total number of vehicles is conserved. Examples of increasing of the speed could be descents. As above all initial data are constant.

More realistic models may be obtained by localizing along the road many source terms as those considered here above.

5 Proof of Theorem 2.2 and Further Estimates

In this section we provide those extensions to the proof of [7, Theorem 2.2] necessary to obtain a proof of the present Theorem 2.2. Indeed, the situation considered in [7] is obtained from the present case setting $A = 0$ and strengthening the assumption *(ii)* on the source s. We use the notations in [7], in particular total variations and $\mathbf{L}^\infty$ norms are all computed through the Riemann invariants w, see [7, (4.1)].

A straightforward extension of Theorem 2.2 in [7] is obtained by replacing the assumptions $0 \in \Omega$, $w(0) = 0$, $s(t, x, 0) = 0$ made there with $w(\underline{u}) = 0$, $s(t, x, \underline{u}) = 0$, $\left|s(t, x, u)\right| \le b(t) \cdot |u - \underline{u}|$, for some $\underline{u} \in \Omega$. By the change of variables $u = \underline{u} + v$ we deduce the existence and uniqueness of an evolution operator $F_{t_o,t}$, for $(t_o, t) \in \mathcal{I}$. Moreover for any positive M and T there exists a constant C such that for any initial data $u_o \in \underline{u} + \mathcal{D}_M$ and $(t_o, t) \in \mathcal{I}_T$ we have the estimates

$$\begin{aligned} \left\|F_{t_o,t}u_o\right\|_\infty &\le \left\|\underline{u}\right\|_\infty + e^{C\int_{t_o}^{t} b(\tau)\,d\tau} \cdot \left\|u_o - \underline{u}\right\|_\infty \\ \mathrm{TV}\left(F_{t_o,t}u_o\right) &\le e^{C(t-t_o)}\left(\mathrm{TV}(u_o - \underline{u}) + C \cdot \mu(\mathbb{R}) \cdot (t - t_o)\right) . \end{aligned}$$

Toward a proof of Theorem 2.2 consider the approximate problem

$$\partial_t u = s^\varepsilon(t, x, u) \tag{5.1}$$

where

$$s^\varepsilon(t, x, u) = \sum_{k\in\mathbb{Z}} \frac{1}{\varepsilon}\left(\int_{(k-1)\varepsilon}^{k\varepsilon} s(t, \xi, u)\,d\xi\right) \cdot \chi_{\left](k-1)\varepsilon,\, k\varepsilon\right]}(x) \tag{5.2}$$

Then, [7, Lemma 4.2] is replaced by the following entirely similar result that applies also to the more general estimate (2.10).

Lemma 5.1 *Let s be as in* (ii). *Then also s^ε satisfies* (ii) *with the same L_K in* (2.9) *and A, B in* (2.10). *The estimate* (2.8) *holds if modified as follows: if $h, k \in \mathbb{Z}$ and $h \leq k$, for $x_1 \in \left]h\varepsilon, (h+1)\varepsilon\right]$ and $x_2 \in \left]k\varepsilon, (k+1)\varepsilon\right]$, then*

$$\left|s^\varepsilon(t, x_1, u) - s^\varepsilon(t, x_2, u)\right| \leq 3\mu\left(\left[h\varepsilon, (k+1)\varepsilon\right]\right) . \tag{5.3}$$

The domain $\tilde{\Omega} \ni 0$ is the Cartesian product of closed, possibly unbounded, intervals, i.e. $\tilde{\Omega} = \prod_{i=1}^n \mathcal{I}_i$, see [13]. We assume that for any $\varepsilon > 0$ there exist scalar ε-grids $\mathcal{I}_1^\varepsilon \subset \mathcal{I}_1, \ldots, \mathcal{I}_n^\varepsilon \subset \mathcal{I}_n$ such that for every $i = 1, \ldots, n$:

(i) $]w - \varepsilon, w + \varepsilon[\cap \mathcal{I}_i^\varepsilon \neq \emptyset$ for any $w \in \mathcal{I}_i \cap \left[-1/\varepsilon\,, 1/\varepsilon\right]$;

(ii) there exists a positive δ^ε such that $\min\left\{\left|w' - w''\right| : w', w'' \in \mathcal{I}_i^\varepsilon\right\} > \delta^\varepsilon$;

(iii) $\min \mathcal{I}_i^\varepsilon = \begin{cases} -1/\varepsilon \text{ if } \inf \mathcal{I}_i = -\infty \\ \inf \mathcal{I}_i \text{ if } \inf \mathcal{I}_i \in \mathcal{I}_i \end{cases}$ $\max \mathcal{I}_i^\varepsilon = \begin{cases} 1/\varepsilon \quad \text{if } \sup \mathcal{I}_i = \infty \\ \sup \mathcal{I}_i \text{ if } \sup \mathcal{I}_i \in \mathcal{I}_i. \end{cases}$

We shall refer to $\mathcal{G}^\varepsilon = w^{-1}\left(\prod_{i=1}^n \mathcal{I}_i^\varepsilon\right)$ as an ε-grid. We define then

$$\mathbf{X}(\mathcal{G}^\varepsilon) = \left\{u \in \mathbf{X} : u \in \mathbf{PC} \text{ and } u(\mathbb{R}) \subseteq \mathcal{G}^\varepsilon\right\}$$

where **PC** is the set of piecewise constant functions with finitely many jumps. Lemma 4.3 in [7] is replaced by the following lemma.

Lemma 5.2 *Equation* (5.1) *generates a process $\Sigma^\varepsilon : \mathcal{I} \times \mathbf{X} \mapsto \mathbf{X}$ such that $t \mapsto \Sigma^\varepsilon_{t_o,t} u$ is a solution to* (5.1) *with initial datum u at time t_o.*

For all $R > 0$ and $T > t_o$ there exist positive $\tilde{B}, \tilde{L} \in \mathbf{L}^\mathbf{1}_{\mathrm{loc}}\left([t_o, +\infty[; \mathbb{R}\right)$, $d \in \mathbf{W}^{\mathbf{1},\mathbf{1}}\left([t_o, +\infty[\,; \mathbb{R}\right)$ with $d(t_o) = 0$ and a constant $C > 0$, all of them independent from ε, such that for all $t \in [t_o, T]$ and $u \in \mathbf{X}$ with $\|u\|_\infty \leq R$,

$$\begin{aligned} \left\|\Sigma^\varepsilon_{t_o,t} u\right\|_\infty &\leq e^{\int_{t_o}^t \tilde{B}(\tau)\, d\tau} \cdot \left(d(t) + \|u\|_\infty\right) && (5.4) \\ \mathrm{TV}\left(\Sigma^\varepsilon_{t_o,t} u\right) &\leq e^{\int_{t_o}^t \tilde{L}(\tau)\, d\tau} \cdot \left(\mathrm{TV}(u) + C \cdot \mu(\mathbb{R}) \cdot (t - t_o)\right) . && (5.5) \end{aligned}$$

Moreover, given an ε-grid $\mathcal{G}^\varepsilon$, there exists an ε-grid $\langle\mathcal{G}^\varepsilon\rangle$ such that for any $u \in \mathbf{X}(\mathcal{G}^\varepsilon)$, $\Sigma^\varepsilon_{t_1,t_2} u \in \mathbf{X}\left(\langle\mathcal{G}^\varepsilon\rangle\right)$.

Proof. For all $x \in \mathbb{R}$, let $t \mapsto \Sigma^\varepsilon_{t_o,t} u(x)$ be the solution to the Cauchy problem (5.1) with initial datum $u(t_o, x)$ assigned at time t_o. Assumption *(ii)* in Theorem 2.2 allows to apply the standard theory of o.d.e.s ensuring that Σ^ε is well defined and maps $\mathcal{I} \times \mathbf{X}$ into $\mathbf{X}$.

Denote by K a compact set such that the solutions to (2.6), with data u_o satisfying $\|u_o\|_\infty \leq R$, are valued in K for all $t \in [t_o, T]$. Let $\tilde{K} = w(K)$, $L_w = \|D_u w\|_{\mathbf{L}^\infty(K)}$ and $L_u = \|D_w u\|_{\mathbf{L}^\infty(\tilde{K})}$. In Riemann coordinates equation (5.1) reads

$$\partial_t w = \tilde{s}^\varepsilon(t, x, w) \,. \tag{5.6}$$

We prove first $\tilde{s}^\varepsilon$ satisfies estimates analogous to (2.8), (2.9), (2.10). In fact, (2.8) holds as in (5.3), with $L_w \mu$ in place of μ.

Concerning (2.9), for any $w_1, w_2 \in \tilde{K}$, $t \in [t_o, T]$ and a.e. $x \in \mathbb{R}$,

$$\begin{aligned} & \left|\tilde{s}^\varepsilon(t,x,w_2) - \tilde{s}^\varepsilon(t,x,w_1)\right| \\ = \; & \left|Dw(u_2) s^\varepsilon(t,x,u_2) - Dw(u_1) s^\varepsilon(t,x,u_1)\right| \\ \leq \; & \left(\sup_{u\in K} \left|D^2_u w(u)\right|\right) \cdot \left(A(t) + B(t) \sup_{u \in K} |u|\right) \cdot |u_2 - u_1| \\ & \quad + c \cdot L_w \cdot L_K(t) \cdot |u_2 - u_1| \\ \leq \; & \tilde{L}_K(t) \cdot |w_2 - w_1| \end{aligned} \tag{5.7}$$

where $\tilde{L}_K(t) = L_u \cdot \Big(\sup_{u\in K} \left|D^2_u w\right| \cdot \big(A(t) + B(t) \sup_{u\in K} |u|\big) + L_w \cdot L_K(t)\Big)$ and we used the standard $\mathbf{L}^\infty$ norm in $\mathbb{R}$.

Concerning (2.10), from (5.3) and since $0 \in \tilde{\Omega}$ we find

$$\begin{aligned} \left|\tilde{s}^\varepsilon(t,x,w)\right| & \leq |D_u w| \cdot \left|s^\varepsilon(t,x,u)\right| \\ & \leq |D_u w| \cdot \big(A(t) + B(t)|u|\big) \\ & \leq c \cdot L_w \cdot \left(A(t) + B(t) \left(\sup_{u\in K} |u| + L_u \cdot |w|\right)\right) \\ & \leq \tilde{A}(t) + \tilde{B}(t) \big(1 + |w|\big) \end{aligned} \tag{5.8}$$

where $\tilde{A}(t) = c \cdot L_w \cdot A(t)$, for a suitable $c > 0$, and $\tilde{B}(t) = c \cdot L_w \cdot \max\left\{\sup_{u\in K} |u|, L_u\right\} \cdot B(t)$. Note that $\tilde{B} \in \mathbf{L}^1_{\mathrm{loc}}([t_o, +\infty[; \mathbb{R})$ depends on R and then on K.

Consider now (5.4). From (5.8) we have

$$\left\|\Sigma^\varepsilon_{t_o,t} u\right\|_\infty \;\leq\; \|u\|_\infty + \int_{t_o}^t \left(\tilde{A}(\tau) + \tilde{B}(\tau) + \tilde{B}(\tau) \cdot \left\|\Sigma^\varepsilon_{t_o,\tau} u\right\|_\infty\right) d\tau \,.$$

By Gronwall Lemma, (5.4) follows with $d(t) = \int_{t_o}^t \Big(\tilde{A}(\tau) + \tilde{B}(\tau)\Big)\, d\tau$.

Consider (5.5). If $u \in \mathcal{D}$ with $\|u\|_\infty \leq R$, then (5.4) implies that for $t \in [t_o, T]$, the solution $w(t) = w\left(\Sigma^\varepsilon_{t_o,t} u\right)$ to (5.6) with $w(t_o, x) = w(u)$ has values in the compact set $\tilde{H} = w(\Omega) \cap \left[-R_1 \cdot e^{\int_{t_o}^t \tilde{B}(\tau)\, d\tau}, R_1 \cdot e^{\int_{t_o}^t \tilde{B}(\tau)\, d\tau}\right]^n$

for $R_1 = R + \|d\|_{\mathbf{L}^\infty([t_o,T],\mathbb{R})}$. Define $H = u(\tilde H)$. If $h, k \in \mathbb{Z}$ and $h \leq k$, then for all $x_1 \in \left]h\varepsilon, (h+1)\varepsilon\right]$ and $x_2 \in \left]k\varepsilon, (k+1)\varepsilon\right]$ we have for $i = 1, \ldots, n$

$$\begin{aligned}\left|w_i(t,x_2) - w_i(t,x_1)\right| \;\leq\; & \left|w_i(t_o,x_2) - w_i(t_o,x_1)\right| \\ & + \int_{t_o}^{t} \left|\tilde s_i^\varepsilon\left(\tau, x_2, w(\tau,x_2)\right) - \tilde s_i^\varepsilon\left(\tau, x_1, w(\tau,x_2)\right)\right| d\tau \\ & + \int_{t_o}^{t} \left|\tilde s_i^\varepsilon\left(\tau, x_1, w(\tau,x_2)\right) - \tilde s_i^\varepsilon\left(\tau, x_1, w(\tau,x_1)\right)\right| d\tau\,.\end{aligned} \tag{5.9}$$

If $h = k$, from (5.7), setting $\tilde L = \tilde L_H$,

$$\begin{aligned}& \left|w_i(t,x_2) - w_i(t,x_1)\right| \\ \leq\; & \left|w_i(t_o,x_2) - w_i(t_o,x_1)\right| + \int_{t_o}^{t} \tilde L(\tau)\left|w(\tau,x_2) - w(\tau,x_1)\right| d\tau\,.\end{aligned}$$

If $h < k$,

$$\begin{aligned}\left|w_i(t,x_2) - w_i(t,x_1)\right| \;\leq\; & \left|w_i(t_o,x_2) - w_i(t_o,x_1)\right| \\ + \; & \int_{t_o}^{t} \left[3L_w \cdot \mu\left(\left[h\varepsilon, (k+1)\varepsilon\right]\right) + \tilde L(\tau)\left|w(\tau,x_2) - w(\tau,x_1)\right|\right] d\tau\end{aligned}$$

because of (5.3). By Gronwall Lemma we then obtain

$$\begin{aligned}\left|w_i(t,x_2) - w_i(t,x_1)\right| \;\leq\; & e^{\int_{t_o}^{t} \tilde L(\tau) d\tau} \times \\ \times \; & \left(\left|w_i(t_o,x_2) - w_i(t_o,x_1)\right| + 3L_w \cdot \mu\left(\left[h\varepsilon, (k+1)\varepsilon\right]\right) \cdot (t - t_o)\right)\end{aligned}$$

where the term $3L_w \cdot \mu\left(\left[h\varepsilon, (k+1)\varepsilon\right]\right) \cdot (t - t_o)$ is missing if $h = k$. The rest of the proof is as in [7]. $\square$

The proof of Theorem 2.2 now follows through the operator splitting technique, along the same lines in [7].

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References

[1] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.

[2] P. Bagnerini, R. M. Colombo, A. Corli, and S. Pedretti. Conservation versus balance laws in traffic flow. In *Traffic and Granular Flow '03*.

[3] S. Bianchini. The semigroup generated by a Temple class system with non-convex flux function. *Differential Integral Equations*, 13(10-12):1529–1550, 2000.

[4] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[5] R. M. Colombo. A 2 × 2 hyperbolic traffic flow model. *Math. Comput. Modelling*, 35(5-6):683–688, 2002. Traffic flow—modelling and simulation.

[6] R. M. Colombo. Hyperbolic phase transitions in traffic flow. *SIAM J. Appl. Math.*, 63(2):708–721, 2002.

[7] R. M. Colombo and A. Corli. On a class of hyperbolic balance laws. *J. Hyperbolic Differ. Equ.*, 1(4):725–745, 2004.

[8] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2000.

[9] C. F. Daganzo. Requiem for high-order fluid approximations of traffic flow. *Trans. Res.*, 29B(4):277–287, August 1995.

[10] J. M. Greenberg, A. Klar, and M. Rascle. Congestion on multilane highways. *SIAM J. Appl. Math.*, 63(3):818–833 (electronic), 2003.

[11] R. Haberman. *Mathematical models*, volume 21 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1998.

[12] D. Helbing, A. Hennecke, V. Shvetsov, and M. Treiber. Micro- and macro-simulation of freeway traffic. *Math. Comput. Modelling*, 35(5-6):517–547, 2002.

[13] D. Hoff. Invariant regions for systems of conservation laws. *Trans. Amer. Math. Soc.*, 289(2):591–610, 1985.

[14] B. S. Kerner. Phase transitions in traffic flow. In D. Helbing, H. Hermann, M. Schreckenberg, and W. D.E., editors, *Traffic and Granular Flow '99*, pages 253–283. Springer Verlag, 2000.

[15] B. L. Keyfitz and H. C. Kranzer. A system of nonstrictly hyperbolic conservation laws arising in elasticity theory. *Arch. Rational Mech. Anal.*, 72(3):219–241, 1979/80.

[16] C. Lattanzio and P. Marcati. The zero relaxation limit for the hydrodynamic Whitham traffic flow model. *J. Differential Equations*, 141(1):150–178, 1997.

[17] M. J. Lighthill and G. B. Whitham. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.*, 229:317–345, 1955.

[18] H. J. Payne. Models of freeway traffic and control. In G. Bekey, editor, *Mathematical Models of Public Systems, Vol I*, pages 51–61. Simulation Council, 1971.

[19] P. I. Richards. Shock waves on the highway. *Operations Res.*, 4:42–51, 1956.

[20] B. Temple. Systems of conservation laws with invariant submanifolds. *Trans. Amer. Math. Soc.*, 280(2):781–795, 1983.